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GENERALIZED SUBTOUR ELIMINATION CONSTRAINTS AND CONNECTIVITY CONSTRAINTS

by

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ABSTRACT

This paper examines some properties of generalized subtour elimination constraints and connectivity constraints used in a variety of routing problems.

RESUME

Dans cet article on étudie quelques propriétés des contraintes d'élimination de sous-tours et des contraintes de connectivité présentes dans plusieurs problèmes de construction de tournées.

Introduction

Several problems in connection with graphs can be formulated as integer linear programs (ILPs) possessing structural similarities. These include extensions of the well known travelling salesman problem (TSP) and a whole family of vehicle routing problems [1]. Our aim is to show how a class of constraints included the TSP formulation can be generalized. The results are applied to the formulation of three TSP extensions having different characteristics.

The travelling salesman problem

The travelling salesman problem has been studied extensively over the last three decades. It can be defined as follows. Let $G = (N, A, C)$ be a graph where $N = \{1, \ldots, n\}$ is the set of nodes, $A = \{(i,j) : i,j \in N, i \neq j\}$ is the set of edges and $C = (c_{ij})$ is a cost or distance matrix associated with the edges. (For the sake of simplicity, we only consider problems in which C is symmetrical, i.e. $c_{ij} = c_{ji}$ for all i,j; the definition of A can then be restricted to pairs (i,j) for which $i < j$. The transposition to the asymmetrical case of all results derived in this paper is relatively simple.) The TSP consists of determining the shortest cycle passing through every node of N once and only once.

Dantzig et al. [2] were among the first to propose an ILP formulation for the TSP. We present it here with a somewhat different notation which we will then use in the remainder of this paper. Define

x_{ij} $(i < j)$: the number of times edge (i,j) appears in the optimal solution;

$S \subseteq N$ and $\bar{S} = N - S$;

$I(S) = \sum_{i,j \in S} x_{ij}$, the sum of internal edges of S ($|S| \geq 2$);

$E(S) = \sum_{\substack{i \in S, j \in \bar{S} \\ \text{or } i \in \bar{S}, j \in S}} x_{ij}$, the cut-set of S ($|S| \geq 1$);

$d_i = E(\{i\})$, the degree of node i ($i \in N$).

The TSP formulation is then:

(TSP) minimize $\sum_{i,j \in N} c_{ij} x_{ij}$

subject to

(1) $d_i = 2$ ($i \in N$)

(2) $I(S) \leq |S| - 1$ ($S \subset N$; $3 \leq |S| \leq n-3$)

(3) $x_{ij} = 0, 1$ ($i,j \in N$)

Constraints (1) specify the degree of each node. Constraints (2) are *subtour elimination constraints*: their presence ensures that the solution contains no Hamiltonian cycle over less than n nodes. Constraints (3) set the integrality requirements on the variables. These three types of constraints are present in several TSP extensions. This paper concentrates on subtour elimination constraints and on their generalization.

The following equality holds as long as $d_i = 2$ for all $i \in S$:

(4) $2\, I(S) + E(S) = 2|S|$ ($S \subset N$; $2 \leq |S| \leq n-2$)

It is therefore easy to infer from (4) that constraints (2) are equivalent to the following *connectivity constraints*:

$$(5) \quad E(S) \geq 2 \qquad (S \subset N;\ 2 \leq |S| \leq n-2)$$

Connectivity constraints simply state that any set of at least two nodes has to be linked to its complement. In theory, whether constraints (2) or (5) are used is immaterial. However, from a computational point of view, it is preferable to use the form involving the least number of terms.

Finally note that condition (3) is not required for the validity of (2) or (5). These constraints can therefore be used to break any illegal connected component, i.e. subgraphs $G' = (S, A', C')$ of G for which $S \subset N$, $2 \leq |S| \leq n-2$, $0 \leq x_{ij} \leq 1$ for all $(i,j) \in A'$ and $E(S) = 0$.

The capacitated vehicle routing problem

Nobert [3] observed that if $d_i = 2$ for all $i \in S$, the following equivalence can also be derived from (4):

$$(6) \quad I(S) \geq |S| - r \Leftrightarrow E(S) \geq 2r \qquad (S \subset N;\ 2 < |S| \leq n-2;\ r \in R)$$

These stronger subtour elimination constraints and connectivity constraints can be used in the solution of some VRPs, for example, the capacitated vehicle routing problem (CVRP). Consider a graph G as above and associate with each node of N-{1} a positive weight w_i. There are m vehicles, each having the same capacity W. In addition, there is a *depot* located at node 1. The CVRP consists of establishing minimum distance vehicle routes in such a way that (i) each vehicle starts and ends its journey at the depot; (ii) every node, with the exception of the depot, is visited exactly once by exactly one vehicle; (iii) the sum of weights on any vehicle route does not exceed W.

In addition to the notation already introduced, define

V(S) : a lower bound on the number of vehicles required to visit all nodes of S in the optimal solution

The following ILP formulation for the CVRP is adapted from Laporte and Nobert [7]:

(CVRP) minimize $\sum_{i,j \in N} c_{ij} x_{ij}$

subject to

(7) $d_1 = 2m$

(8) $d_i = 2$ $(i \in N-\{1\})$

(9) $I(S) \geq |S| - V(S)$ $(S \subset N-\{1\};\ 3 \leq |S| \leq n-3)$

(10) $x_{1j} = 0, 1, 2$ $(j \in N-\{1\})$

$x_{ij} = 0, 1$ $(i,j \in N-\{1\})$

In constraints (10), $x_{1j} = 2$ corresponds to a return trip between the depot and j. Also note that in constraint (7), m can be considered as a decision variable or as a known constant.

The validity of (9) is easily established: since at least V(S) vehicles are required to service S in the optimal solution, it follows that $E(S) \geq 2V(S)$. By setting $r = V(S)$ in (6), this is equivalent to (9).

In Laporte and Nobert [4], the value of V(S) was set equal to the smallest integer greater than or equal to $\sum_{i \in S} w_i/W$. Gavish [5] later suggested the use of a sharper bound: define V(S) as the optimal number of bins in the solution of the bin packing problem [6] associated with S. Preliminary results [7] indicate that the bin packing bound is particularly effective when the w_i's follow a normal distribution of mean $\mu = \frac{1}{2} W$.

It should finally be mentioned that constraints (9) eliminate connected components disjoint from the depot as well as components containing it, but whose sum of weights exceeds W.

A further generalization

There are important classes of problems for which (6) does not hold. Two of these will be examined in the next sections. First define

$$D(S) = \sum_{i \in S} d_i$$

Equivalence (6) is invalid if either $D(S) = 2|S|$ or if $D(S) < 2|S|$. Indeed, consider the following identity:

(11) $2\, I(S) + E(S) = D(S)$ $\qquad (S \subset N;\ 2 < |S| \leq n-2)$

Then, (i) if $D(S) = 2|S|$, $E(S) \geq 2r \Rightarrow I(S) \geq |S| - r$; (ii) similarly, if $D(S) \geq 2|S|$, $I(S) \geq |S| - r \Rightarrow E(S) \geq 2r$.

However, it is straightforward to conclude from (11) that the following equivalence is valid, even if $D(S) \neq 2|S|$:

(12) $I(S) \geq \frac{1}{2} D(S) - r \Leftrightarrow E(S) \geq 2r$ $\qquad (S \subset N;\ 2 \leq |S| \leq n-2;\ r \in \mathbb{R})$

We now provide two examples of problems for which the equality $D(S) = 2|S|$ does not necessarily hold. In the first case, $D(S) = 2|S|$ for all $S \subseteq N$; in the second case, $D(S) = 2|S|$ for all $S \subseteq N-\{1\}$.

The single depot location-routing problem

Location-routing problems involve the simultaneous location of one or several depots among a set of candidate nodes and the determination of the associated delivery routes. The study of these problems constitutes a fast growing area of location theory. Consider the simple case first treated by Laporte and Nobert [8]. The problem consists of locating one depot at one of several candidate locations in a subset K of N and of constructing delivery routes for a fixed number m of vehicles in such a way that conditions (i) and (ii) of the CVRP hold. Define

y_i : a 0-1 variable taking the value 1 if and only if the depot is located at node i in the optimal solution

The problem can then be formulated as follows:

(LR) minimize $\sum_{i,j \in N} c_{ij} x_{ij}$

subject to

(13) $\sum_{i \in K} y_i = 1$

(14) $d_i = 2 + 2(m-1) y_i \qquad (i \in K)$

(15) $d_i \quad 2$ $\qquad (i \in N\text{-}K)$

(16) $I(S) \cdot \tfrac{1}{2} D(S) - 1$ $\qquad (S \subset N;\ 2 \le |S| \le n\text{-}2)$

(17) $y_i = 0, 1$ $\qquad (i \in K)$

(18) $x_{ij} = 0, 1, 2$ $\qquad (i \text{ or } j \in K)$

$x_{ij} = 0, 1$ $\qquad (i,j \in N\text{-}K)$

All constraints of this model are self-explanatory. Note that (16) is equivalent to the following constraint used in Laporte and Nobert [8]:

(19) $I(S) \le |S| - 1 + (m-1) \sum_{i \in S \cap K} y_i$ $\qquad (S \subset N;\ 2 \le |S| \le n\text{-}2)$

This equivalence is proved by considering the following two cases: (i) if in the optimal solution the depot belongs to S, then $\sum_{i \in S \cap K} y_i = 1$ and $D(S) = 2|S| + 2m - 2$; (ii) otherwise, $\sum_{i \in S \cap K} y_i = 0$ and $D(S) = 2|S|$.

We now propose a formulation for the case where m is a variable bounded above by $\bar{m}$. This case is of interest only if the set of potential depots contains only nodes i for which $c_{jk} > c_{ij} + c_{ik}$ for some j and k; if there exists no such node i in K, there will only be one vehicle in the optimal solution and (LR) then reduces to (TSP). Since m is now a variable, some transformations are necessary in order to remove the product term in each of constraints (14). For this, replace these constraints by (20) and (21):

(20) $d_i = \delta_i \cdot 2 + 2(\bar{m}-1)\, y_i$ $\qquad (i \in K)$

(21) $v_i = 2, \ldots, 2\bar{m}$ $\qquad (i \in K)$

It is not necessary to transform constraints (16) which do not contain the factor m, but this would not have been the case if (19) had been used in (LR).

The orienteering problem

The orienteering problem[9,16] (OP) derives its name from an outdoor treasure hunt game played in some Scandinavian countries. In the OP, there is a positive profit p_i associated with every node i of N-{1}. The problem consists of determining a Hamiltonian cycle of maximum profit (i) passing through node 1 and (ii) having a total length not exceeding a prespecified bound L. We propose the following formulation for the OP:

(OP) maximize $\frac{1}{2} \sum_{i \in N-\{1\}} p_i d_i$

subject to

(22) $\sum_{i,j \in N} c_{ij} x_{ij} \leq L$

(23) $d_1 = 2$

(24) $d_i = v_i$ $\qquad (i \in N-\{1\})$

(25) $I(S) \leq D(S) \, (|S|-1)/2|S|$ $\qquad (S \subset N-\{1\};\ 2 \leq |S| \leq n-2)$

(26) $x_{1j} = 0, 1, 2$ $\qquad (j \in N-\{1\})$

$x_{ij} = 0, 1$ $\qquad (i,j \in N-\{1\})$

(27) $\delta_i = 0, 2 \qquad (i \in N-\{1\})$

Subtour elimination constraints (25) prohibit the occurrence of subtours disconnected from node 1. Before deriving them, note that it would have been valid to impose $I(S) \leq |S| - 1$ since this constraint clearly eliminates any connected component disjoint from node 1 and involving all nodes of S. However, it does not prevent situations in which some nodes of S form a connected component disjoint from node 1 while the others do not appear in the solution. Formally, it does not eliminate solutions for which $S' \subset N-\{1\}$, $I(S') = |S'|$, $E(S') = 0$ and $D(S-S') \neq 0$. On the other hand, the tighter form $I(S) \leq \frac{1}{2} D(S) - 1$ would lead to an infeasible ILP whenever $D(S) = 0$ and, for the same reason, connectivity constraints $E(S) \geq 2$ are also too strong.

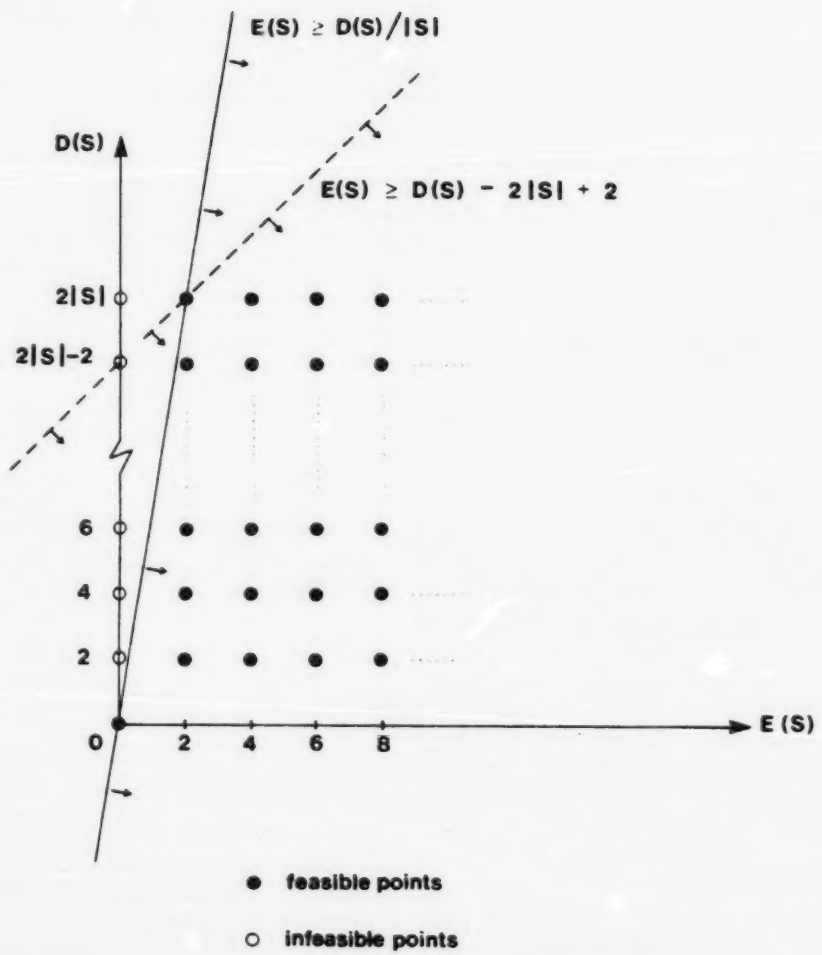
Constraints (25) can be derived as follows. Consider a non-empty subset S of $N-\{1\}$. In the optimal solution, either (i) $E(S) = 0$ and $D(S) = 0$ or (ii) $E(S) \geq 2$ and $2 \leq D(S) \leq 2|S|$. The feasible region corresponding to this dichotomy is depicted by figure 1. The optimum must lie below the straight line passing through (0,0) and (2, 2|S|), i.e. the following constraints must be satisfied:

(28) $E(S) \geq D(S)/|S| \qquad (S \subset N-\{1\};\ 2 \leq |S| \leq n-2)$

Constraints (25) are then easily obtained by combining (11) and (28).

Finally note that the weaker subtour elimination constraint $I(S) \leq |S| - 1$ is equivalent to $E(S) \geq D(S) - 2|S| + 2$. This constraint corresponds to the dotted line passing through (0, 2|S| - 2) and (2, 2|S|) in figure 1. It eliminates the infeasible point (0, 2|S|) without cutting any feasible solution, but does not eliminate infeasible points of the form (0, D(S)) where $2 \leq D(S) \leq 2|S| - 2$.

FIGURE 1


E(S) ≥ D(S)/|S|
D(S)
E(S) ≥ D(S) − 2|S| + 2
2|S|
2|S|−2
6
4
2
0
2
4
6
8
E (S)
feasible points
infeasible points

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